# Contrasts and the Average Density in the Hierarchical Model of the Universe (An unpublished theorem of Andrzej Zieba)

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The proof of Andrzej Zieba's unpublished theorem on density contrasts in a hierarchical Universe is presented.

Charlier (1908) proposed a hierarchical model of the Universe in order to avoid photometric paradoxes. Although paradoxes can be easily cleared up by the Standard Model or other relativistic models now, the hierarchical structure of the Universe is still a useful construction, especially, when we consider the recent observational evidence testifying against the expansion of the Universe (*e.g. Festschrift Vigier*, 1991).

In the Krakow Extragalactic Astronomy Conversatory in 1974, Andrzej Zieba discussed the cases when the average density of the Universe is not equal to zero. His unpublished theorem about the

Hierarchical Universe is presented here with the reconstruction of a simple proof.

Zieba formulated his theorem as follows:

> *If the average density of an infinite Universe possessing infinite hierarchical structure is not equal to zero then the density contrasts between subsequent hierarchical levels decrease toward zero.*

To prove the theorem we have to explain the meaning of the word "contrast". Let us describe the Universe as consisting of spherical systems that are equal to one another on the same level. The non-spherical systems having different sizes at the same level can be represented by homogeneous spheres of average radius and density. The first-level system is a star, the second-level a galaxy, the third-level, a cluster of galaxies and so on. We define quantities as follows:

$\mathbf{r}_i$ average density of the *i*-th system

$r_i$ average radius of the *i*-th system,

$n_i$ number of bodies: constituents of the (*i* - 1)-th system which belong to *i*-th system.

These values are not independent and we can find the average density of the *i*-th system $\mathbf{r}_i$ as a function of other quantities:

$$\mathbf{r}_2 = \frac{4/3\, p\, r_1^3 n_1 \mathbf{r}_1}{4/3\, p\, r_2^3} = n_1 \mathbf{r}_1 \frac{r_1^3}{r_2^3},$$

$$\mathbf{r}_3 = \frac{4/3\, p\, r_2^3 n_2 \mathbf{r}_2}{4/3\, p\, r_3^3} = n_2 \mathbf{r}_2 \frac{r_2^3}{r_3^3} = n_1 n_2 \mathbf{r}_1 \frac{r_1^3}{r_3^3}$$

Generally:

$$\boldsymbol{r}_i = \boldsymbol{r}_1 \frac{r_1^3}{r_i^3} \prod_{k=1}^{i-1} n_k \tag{1}$$

The average density of the Universe $\boldsymbol{r}$ is defined as follows:

$$\boldsymbol{r} = \lim_{i \Rightarrow \infty} \boldsymbol{r}_i \tag{2}$$

Let us accept the following definition of growing contrasts: The contrasts between systems are defined by

$$B_i = \frac{\boldsymbol{r}_{i-1}}{\boldsymbol{r}_i}, i = 2, 3, 4, \ldots \tag{3}$$

and grow when

$$B_2 < B_3 < \ldots < B_i < \ldots$$

The density of a system cannot be greater than the density of its constituents, so $B_i \geq 1$ Using (1) to the left side of (3) we obtain:

$$B_{i+1} = \frac{\boldsymbol{r}_i}{\boldsymbol{r}_{i+1}} = \frac{r_{i+1}^3}{r_i^3} \frac{1}{n_i}$$

and

$$n_i = \frac{1}{B_{i+1}} \frac{r_{i+1}^3}{r_i^3}$$

using $n_i$ in this formula again in (1) we find

$$\boldsymbol{r}_i = \boldsymbol{r}_1 \frac{r_1^3}{r_i^3} \frac{1}{B_2} \frac{r_2^3}{r_1^3} \frac{1}{B_3} \frac{r_3^3}{r_2^3} \cdots \frac{1}{B_i} \frac{r_i^3}{r_{i-1}^3}$$

thus

$$\boldsymbol{r}_i = \boldsymbol{r}_1 \frac{1}{B_2 B_3 \ldots B_i} \tag{4}$$

growing contrasts mean that $B_{i+1} > B_i$, but we know that $B_i \geq 1$; so for growing contrasts

$$\lim_{i \Rightarrow \infty} \left[ B_2 B_3 .. B_i \right] = +\infty$$

then $r = 0$.

Examining other cases, we may treat Zieba's theorem as follows:

1) If contrasts between subsequent systems increase then the average density of the Universe is equal to zero.
2) If contrasts are the same and equal to 1, then $r = r_1$ (totally homogeneous Universe - any hierarchy practically vanishes).
3) If all the contrasts are the same but different from 1 then $r = 0$.
4) If contrasts decrease, then this criterion does not work. We can give two examples for point 4):

a) $B_i = \exp\left( 1/2^i \right)$

$$\ln\left[ \lim_{i \Rightarrow \infty} \left( B_2 B_3 .. B_i \right) \right] = \lim_{i \Rightarrow \infty} \left( \ln B_1 + \ln B_2 + .. + \ln B_i \right)$$

$$= \lim_{i \Rightarrow \infty} \left( 1/4 + 1/8 + .. + 1/2^i \right) = \frac{1}{2}$$

$$\lim_{i \Rightarrow \infty} B_2 B_3 .. B_i = e^{1/2}$$

$$r = \frac{r_1}{e^{1/2}}$$

b) $B_i = \exp\left( 1/i \right)$

$$\ln\left[ \lim_{i \Rightarrow \infty} \left( B_2 B_3 .. B_i \right) \right] = \lim_{i \Rightarrow \infty} \left( 1/2 + 1/3 + .. + 1/i \right) = +\infty$$

$$r = 0$$

In both cases a) and b) the contrasts decrease, but the average density of the Universe is equal to zero in one case only. The examples show that in point 4) we have to examine the sequence

$$\ln B_2, \ln B_3, .., \ln B_i, ..$$

If this sequence is convergent then $r \neq 0$, otherwise $r = 0$.

We have therefore proved that the necessary condition for an infinite Universe with infinite hierarchical structure and mean density not equal to zero is a vanishing contrast ("homogenization") with $i$ tending toward $\infty$. But this condition is not sufficient.

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## References:

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